

On the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}, \sigma(3^b) = p^{f_2}q^{g_2}, \sigma(5^c) = p^{f_3}q^{g_3*†}$

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Abstract

We shall solve the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}, \sigma(3^b) = p^{f_2}q^{g_2}, \sigma(5^c) = p^{f_3}q^{g_3}$ with p, q distinct primes.

1 Introduction

As usual, let $\sigma(N)$ denote the sum of divisors of N and $\omega(N)$ the number of distinct prime factors of N . In [18], the author has shown that there are only finitely many odd superperfect numbers (i.e. the number satisfying $\sigma(\sigma(N)) = 2N$) with bounded number of distinct prime factors by proving that the simultaneous equation $\sigma(p_i^{e_i}) = q_1^{f_{1i}} \cdots q_k^{f_{ki}}$ for $k+1$ prime powers $p_i^{e_i} (i = 1, 2, \dots, k+1)$ cannot have solutions with p_1, \dots, p_{k+1} all small. In this paper, we use the method developed in [18] to solve the simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}, \sigma(3^b) = p^{f_2}q^{g_2}, \sigma(5^c) = p^{f_3}q^{g_3}$ with p, q distinct primes.

Wakulicz [16] has shown that all solutions of the purely exponential diophantine equation $2^n - 5^m = 3$ are $(n, m) = (2, 0), (3, 1)$ and $(7, 3)$, from which Makowski and Schinzel [9] derived that the equation $\sigma(2^a) = \sigma(5^c)$ has only the solution $(a, c) = (4, 2)$. We note that it is easy to show that $\sigma(2^a) = \sigma(3^b)$ has no nontrivial solution and $\sigma(3^b) = \sigma(5^c)$ also has no nontrivial solution. Bugeaud and Mignotte [3] has shown that neither of $\sigma(2^a), \sigma(3^b), \sigma(5^c)$ can be perfect power except $\sigma(3) = 2^2$ and $\sigma(3^4) = 11^2$. Moreover, they have shown that the only perfect powers $\frac{x^n-1}{x-1}$ with $x = z^t, z \leq 10$ are $(3^5 - 1)/2 = 11^2$ and $(7^4 - 1)/6 = 20^2$.

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Now we shall state our result.

Theorem 1.1. *The simultaneous equations $\sigma(2^a) = p^{f_1}q^{g_1}$, $\sigma(3^b) = p^{f_2}q^{g_2}$, $\sigma(5^c) = p^{f_3}q^{g_3}$ with $a, b, c > 0$, $f_1, f_2, f_3, g_1, g_2, g_3 \geq 0$ and p, q distinct primes has only the following solutions: i) $(a, b, c) = (1, 1, 1)$. ii) $(a, b, c) = (4, 1, 2)$, iii) $(a, b, c) = (4, 4, 2)$ and iv) $(a, c) = (4, 2)$ and $\sigma(3^b)$ is prime. In other words, if $\omega(\sigma(2^a 3^b 5^c)) \leq 2$, then (a, b, c) must satisfy one of the above.*

Our result is related to the Nagell-Ljunggren equation

$$\frac{x^m - 1}{x - 1} = y^n, x \geq 2, y \geq 2, m \geq 3, n \geq 2, \quad (1)$$

which has been conjectured to have only three solutions $(x, y, m, n) = (3, 11, 5, 2), (7, 20, 4, 2), (18, 7, 3, 3)$. Some of recent remarkable results concerning to the Nagell-Ljunggren equation are [2], [3], [11], [12] and [4]. Our result leads us to conjecture that the diophantine equation

$$\frac{x^l - 1}{x - 1} = y^m z^n \quad (2)$$

has only finitely many solutions in integers $x \geq 2, z \geq y \geq 2$ and $l, m, n \geq n_0$ for some constant n_0 . The *abc*-conjecture, which Mochizuki [13] claims to prove, would allow us to take $n_0 = 3$. Indeed, applying the *abc*-conjecture to the equation $1 + (x - 1)y^m z^n = x^l$, we see that for any given $\epsilon > 0$, the inequality

$$\frac{2}{l} + \frac{l - 1}{l} \times \frac{2}{n + \min\{n, m\}} \geq 1 - \epsilon \quad (3)$$

would hold for sufficiently large x^l . Hence, with only finitely many exceptions, we would have i) $l \geq 5, n = 1$, ii) $l = 4, n = 1$, iii) $(l, m, n) = (4, 1, 2)$. iv) $l = 3, n \leq 2$ or v) $(l, m, n) = (3, 1, 3)$.

2 Preliminary Lemmas

In this section, we introduce some preliminary lemmas. One is Matveev's lower bound for linear forms of logarithms [10].

Lemma 2.1. *Let a_1, a_2, \dots, a_n be positive integers such that $\log a_1, \dots, \log a_n$ are not all zero and $A_j \geq \max\{0.16, \log a_j\}$ for each $j = 1, 2, \dots, n$. More-*

over, we put

$$\begin{aligned}
B &= \max\{1, |b_1| A_1/A_n, |b_2| A_2/A_n, \dots, |b_n|\}, \\
\Omega &= A_1 A_2 \dots A_n, \\
C_0 &= 1 + \log 3 - \log 2, \\
C_1(n) &= \frac{16}{n!} e^n (2n+3)(n+2)(4(n+1))^{n+1} \left(\frac{1}{2} en\right) \\
&\quad \times (4.4n + 5.5 \log n + 7)
\end{aligned} \tag{4}$$

and

$$\Lambda = b_1 \log a_1 + \dots + b_n \log a_n. \tag{5}$$

Then we have

$$\log |\Lambda| > -C_1(n)(C_0 + \log B) \max\left\{1, \frac{n}{6}\right\} \Omega. \tag{6}$$

The others concern to some arithmetical properties of values of cyclotomic polynomials. Lemma 2.2 is a basic and well-known result of this area. Lemma 2.2 has been proved by Zsigmondy [19] and rediscovered by many authors such as Dickson [6] and Kanold [7]. We need only the special case $b = 1$, where this lemma had already been proved by Bang [1]. See also Theorem 6.4A.1 in [14].

Lemma 2.2. *If $a > b \geq 1$ are coprime integers, then $a^n - b^n$ has a prime factor which does not divide $a^m - b^m$ for any $m < n$, unless $(a, b, n) = (2, 1, 6)$, $a - b = n = 1$, or $n = 2$ and $a + b$ is a power of 2.*

Let $o_p(a)$ denote the residual order of $a \pmod{p}$. Lemma 2.2 immediately gives the following result.

Lemma 2.3. *If $(a^e - 1)/(a - 1) = p^{f_1} q^{f_2}$ for some integers a, e, f_1, f_2 and primes $p < q$, then we have $(a, e, p, q, f_1, f_2) = (2, 6, 3, 7, 2, 1)$, $e = r$ or $e = r^2$ for some prime r . Moreover, in the case $e = r$, then we have $p = r, o_q(a) = r$ or $o_p(a) = o_q(a) = r$. In the case $e = r^2$, we have $(p, q, f_1, f_2) = ((a^r - 1)/(a - 1), (a^{r^2} - 1)/(a^r - 1), 1, 1)$ or $(a, e, p, f_1) = (2^m - 1, 4, 2, m + 1)$ for some integer m .*

The following lemma is proved in [3], as mentioned in the introduction.

Lemma 2.4. *Let a, e, x, f be positive integers with $a, x, f > 1$ and $e > 2$. The equation $(a^e - 1)/(a - 1) = x^f$ has no solution but $(a, e, x, f) = (3, 5, 11, 2), (7, 4, 20, 2)$ in integers $2 \leq a \leq 10, e > 2, x > 1, f > 1$.*

Using results mentioned in the introduction, we can immediately solve some special case of our main theorem.

Lemma 2.5. *Choose $a < b$ from the first three primes 2, 3, 5. If $(a^e - 1)/(a - 1) = p^k$ and $(b^f - 1)/(b - 1) = p^l$ for some integers e, f, k, l and some prime p , then $(a^e, b^f) = (2^5, 5^3)$ and $p = 31, k = l = 1$.*

Proof. In the case $k = l = 1$ and $(a^e - 1)/(a - 1) = (b^f - 1)/(b - 1)$, as observed in the introduction, we have $(a^e, b^f) = (2^5, 5^3)$.

Lemma 2.4 yields that the perfect power case must arise from $(3^5 - 1)/2 = 11^2$ or $(3^2 - 1)/2 = 2^2$. In this case, we must have $2^e - 1 = 2$ or 11 or $(5^f - 1)/4 = 2$ or 11 , which is clearly impossible. \square

3 Main Theory

For convenience, we put $a_1 = 2, a_2 = 3, a_3 = 5$ and $e_1 = a+1, e_2 = b+1, e_3 = c+1$.

Lemma 3.1. *For each $i = 1, 2, 3$, we have*

$$e_i \log a_i < E_i = C_i \log p \log q (\log \log p + C_{i+3}), \quad (7)$$

where $C_1 = 1.422 \times 10^{10}, C_2 = 1.226 \times 10^{12}, C_3 = 1.795 \times 10^{12}, C_4 = 23.3, C_5 = 27.8, C_6 = 28.1$.

Proof. Let $\Lambda_i = f_i \log a_i + g_i \log q + \log(a_i - 1) - e_i \log 2 = \log(1 - a_i^{-e_i})$ for $i = 1, 2, 3$. It immediately follows from Matveev's theorem that

$$-\log |\Lambda_1| < C(3) \left(C_0 + \log \frac{e_1 \log 2}{\log q} \right) \log 2 \log p \log q \quad (8)$$

and

$$-\log |\Lambda_j| < C(4) \left(C_0 + \log \frac{e_j \log a_j}{\log q} \right) \log 2 \log a_j \log p \log q \quad (9)$$

for $j = 2, 3$.

Now we shall show (7) in the case $i = 1$. We may assume that $e_1 > 10^{10} \log q / \log 2$. Since $0 < |\Lambda_1| = -\log(1 - 2^{-e_1}) < \frac{1}{2^{e_1-1}}$, we have $-\log |\Lambda_1| >$

$\log(2^{e_1} - 1) > (1 - 10^{-10})e_1 \log 2$. Combining upper and lower bounds for Λ_1 , we obtain

$$\begin{aligned} \frac{e_1 \log 2}{\log q} &< (1 + 10^{-10}) \left(C_0 + \log \frac{e_1 \log 2}{\log q} \right) C(3) \log 2 \log p \\ &< 1.244 \times 10^{10} \log p \log \frac{e_1 \log 2}{\log q}. \end{aligned} \quad (10)$$

Hence, observing that $1.244 \times 10^{10} \log p \geq 1.244 \times 10^{10} \log 2$, we obtain

$$\begin{aligned} \frac{e_1 \log 2}{\log q} &< 1.143 \times (1.244 \times 10^{10} \log p) \log(1.244 \times 10^{10} \log p) \\ &< 1.422 \times 10^{10} (\log \log p + 23.3), \end{aligned} \quad (11)$$

giving (7) in the case $i = 1$.

Next we shall prove (7) in the case $i = 2$. We may assume that $e_2 > 10^{10} \log q / \log 3$ as with the previous case. Since $0 < |\Lambda_2| = -\log(1 - 3^{-e_2}) < \frac{1}{3^{e_2}-1}$, we have $-\log |\Lambda_2| > \log(3^{e_2} - 1) > (1 - 10^{-10})e_2 \log 3$. From $0 < |\Lambda_2| = -\log(1 - 3^{-e_2}) < \frac{1}{3^{e_2}-1}$, we see that $-\log |\Lambda_2| > \log(3^{e_2} - 1) \geq (1 - 10^{-10})e_2 \log 3$ and therefore

$$\begin{aligned} \frac{e_2 \log 3}{\log q} &< (1 + 10^{-10}) \left(C_0 + \log \frac{e_2 \log 3}{\log q} \right) C(4) \log 2 \log 3 \log p \\ &< 1.089 \times 10^{12} \log p \log \frac{e_2 \log 3}{\log q}. \end{aligned} \quad (12)$$

This gives (7) in the case $i = 2$.

Similarly, (7) in the case $i = 3$ follows from

$$\begin{aligned} \frac{e_3 \log 5}{\log q} &< (1 + 10^{-10}) \left(C_0 + \log \frac{e_3 \log 5}{\log q} \right) C(4) \log 2 \log 5 \log p \\ &< 1.595 \times 10^{12} \log p \log \frac{e_3 \log 5}{\log q}. \end{aligned} \quad (13)$$

A similar argument yields (7) in the case $i = 3$. This completes the proof of the lemma. \square

Next, we shall show that we cannot have all of $a_i^{e_i}$'s large.

Lemma 3.2. *Let x be the smallest among $a_i^{e_i}$'s. Let $h_1 = f_2 g_3 - f_3 g_2$, $h_2 = f_3 g_1 - f_1 g_3$ and $h_3 = f_1 g_2 - f_2 g_1$ and $H = \max |h_i|$. Then*

$$\log x \leq \log \frac{7H}{4} + C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5. \quad (14)$$

Proof. We begin by observing that

$$(2^{e_1} - 1)^{h_1} \left(\frac{3^{e_2} - 1}{2} \right)^{h_2} \left(\frac{5^{e_3} - 1}{4} \right)^{h_3} = 1. \quad (15)$$

Now we put

$$\begin{aligned} \Lambda &= (e_1 h_1 - h_2 - 2h_3) \log 2 + e_2 h_2 \log 3 + e_3 h_3 \log 5 \\ &= h_1 \log \frac{2^{e_1}}{2^{e_1} - 1} + h_2 \log \frac{3^{e_2}}{3^{e_2} - 1} + h_3 \log \frac{5^{e_3}}{5^{e_3} - 1}. \end{aligned} \quad (16)$$

Then we have

$$0 < |\Lambda| \leq H \left(\frac{1}{2^{e_1} - 1} + \frac{1}{3^{e_2} - 1} + \frac{1}{5^{e_3} - 1} \right) \leq \frac{7H}{4x} \quad (17)$$

and therefore

$$\log |\Lambda| \leq -\log x + \log \frac{7H}{4}. \quad (18)$$

It follows from the assumption $e_i > 0$ that $\Lambda \neq 0$. Hence Matveev's lower bound gives

$$\log |\Lambda| \geq -C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5. \quad (19)$$

Combining (18) and (19), we obtain (14). \square

The third step is to obtain upper bounds for each e_i .

Lemma 3.3. *Unless $x = p = 31$, we have $e_1 < 4.44 \times 10^{52}$, $e_2 < 2.54 \times 10^{54}$ and $e_3 < 2.55 \times 10^{54}$ and $H < 2.89 \times 10^{68}$.*

Proof. We may assume without the loss of generality that $p < q$. We begin by considering $q \mid x$. In this case, we have

$$\log q < \log x < \log \frac{7H}{4} + C(3)(C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5. \quad (20)$$

We note that $H \leq C_2 C_3 \log p \log q (\log \log p + C_5) (\log \log p + C_6)$ since it follows from Lemma 3.1 that $f_i < e_i \log a_i / \log p_i < C_i \log q (\log \log p + C_{i+3})$ and $g_i < e_i \log a_i / \log q_i < C_i \log p (\log \log p + C_{i+3})$. Hence we obtain $\log p < \log q < 4.35 \times 10^{12}$.

Now we consider the case $p < q$ and $q \nmid x$. Put i to be the index such that $x = (a_i^{e_i} - 1) / (a_i - 1)$, j, k be the others and

$$\begin{aligned} \Lambda' &= e_j h_j \log a_j + e_k h_k \log a_k - h_j \log(a_j - 1) - h_k \log(a_k - 1) + h_i \log x \\ &= h_j \log \frac{a_j^{e_j}}{a_j^{e_j} - 1} + h_k \log \frac{a_k^{e_k}}{a_k^{e_k} - 1}. \end{aligned} \quad (21)$$

It follows from Lemma 2.5 that if $(a_j^{e_j} - 1)/(a_j - 1) = p^{f_j}$ or $(a_k^{e_k} - 1)/(a_k - 1) = p^{f_k}$, then $a_i^{e_i} = 2^5$ or 5^3 and $x = p = 31$. Hence we see that both numbers $(a_j^{e_j} - 1)/(a_j - 1)$, $(a_k^{e_k} - 1)/(a_k - 1)$ must be divisible by q unless $x = p = 31$.

Thus we obtain

$$0 < \Lambda' < H \left(\frac{1}{a_j^{e_j} - 1} + \frac{1}{a_k^{e_k} - 1} \right) \leq \frac{3H}{2q}. \quad (22)$$

As in the previous case, Matveev's theorem now gives

$$\log |\Lambda'| \geq -C(4) \left(C_0 + \log \frac{E_3 H}{\log x} \right) \log 2 \log 3 \log 5 \log x. \quad (23)$$

Combining (22) and (23), we obtain

$$\log q \leq \log \frac{3H}{2} + C(4) \left(C_0 + \log \frac{E_3 H}{\log x} \right) \log 2 \log 3 \log 5 \log x. \quad (24)$$

Since $E_3 = C_3 \log p \log q (\log \log p + C_6) \leq C_3 \log x \log q (\log \log x + C_6)$ and $H < C_2 C_3 (\log q)^2 (\log \log q + C_5) (\log \log q + C_6)$, combining (14) and (24), we obtain $\log q < 3.45 \times 10^{27}$. Moreover, we have

$$\begin{aligned} \log p &= \log x \\ &< \log \frac{7H}{4} + C(3) (C_0 + \log((e_1 + 3)H)) \log 2 \log 3 \log 5 \\ &< 7.22 \times 10^{12}. \end{aligned} \quad (25)$$

So that, we conclude that in both cases, we have $\log p < 7.22 \times 10^{12}$ and $\log q < 3.45 \times 10^{27}$. Now Lemma 3.1 immediately gives that $e_1 < 4.44 \times 10^{52}$, $e_2 < 2.54 \times 10^{54}$ and $e_3 < 2.55 \times 10^{54}$. Finally, the upper bound $H < 2.89 \times 10^{68}$ follows from $H < C_2 C_3 (\log p) (\log q) (\log \log p + C_6) (\log \log q + C_5)$. \square

Now, using the lattice reduction algorithm, we shall obtain feasible upper bounds.

Lemma 3.4. *We have $\log x < 354.8$. Moreover, if $p < q$ and q divides x , then $\log x < 249.5$.*

Proof. We begin by noting that we can assume $x \neq 31$ without the loss of generality.

In order to reduce our upper bounds, we use the LLL lattice reduction algorithm introduced in [8]. Let M be the matrix defined by $m_{12} = m_{13} = m_{21} = m_{23} = 0$ and $m_{11} = m_{22} = \gamma$ and $m_{3i} = \lfloor C\gamma \log a_i \rfloor$ for $i = 1, 2, 3$, where C and γ are constants chosen later. Let L denote the LLL-reduced matrix of M and $l(L)$ the shortest length of vectors in the lattice generated by the column vectors of L .

From the previous lemma, we know that Λ has coefficients with absolute values at most $H \max\{e_1 + 3, e_2, e_3\} < 7.37 \times 10^{122}$. It is implicit in the proof of Lemma 3.7 of de Weger's book [17] that if $l(\Gamma) > X_1 \sqrt{16 + 4\gamma}$ and $X_1 \geq 7.37 \times 10^{122}$, then $|\Lambda| > X_1/(C\gamma)$.

Taking $C = 10^{370}$, $\gamma = 2$, we can confirm that $l(\Gamma) > X_1 \sqrt{16 + 4\gamma}$ and therefore we obtain that $|\Lambda| > 3.685 \times 10^{-248}$. Hence we have

$$\log x < \log \frac{7H}{4} - \log |\Lambda| < 727.94. \quad (26)$$

We choose the index i such that $x = (a_i^{e_i} - 1)/(a_i - 1)$ and let j, k be the others. From the above estimate for x , we derive that

$$e_i \leq \left\lfloor \frac{\log 2x}{\log a_i} \right\rfloor \leq 1051. \quad (27)$$

We consider the case $p < q$ and q does not divide x . From (24) we obtain $\log q < 3.337 \times 10^{17}$. Lemma 3.1 gives that

$$\begin{aligned} |h_i| &< C_2 C_3 \log x \log q (\log \log x + C_6) (\log \log q + C_5) \\ &< 1.264 \times 10^{48}, \end{aligned} \quad (28)$$

$$|h_j| = |f_i g_k| < C_3 \log x (\log \log q + C_6) < 8.944 \times 10^{16}, \quad (29)$$

$$|e_j| < C_3 \log x \log q (\log \log q + C_6) / \log 2 < 4.306 \times 10^{34} \quad (30)$$

and similar upper bounds hold for $|h_k|$ and $|e_k|$, respectively. Hence Λ has coefficients with absolute values at most 3.852×10^{51} . Using the LLL-reduction again with $C = 10^{157}$ and $\gamma = 2$, we obtain $|\Lambda| > 1.926 \times 10^{-106}$ and therefore $\log x < \log(7H/4) - \log |\Lambda| < 354.8$.

Next, we consider the case $p < q$ and q divides x . In this case, we have $\log p < \log q \leq \log x < 727.94$. We choose the index i such that $x = (a_i^{e_i} - 1)/(a_i - 1)$ and let j, k be the others. Lemma 3.1 gives that

$$|h_i| < C_2 C_3 \log^2 x (\log \log x + C_5) (\log \log x + C_6) < 1.392 \times 10^{33}, \quad (31)$$

$$|h_j| \leq \max |f_i g_k, f_k g_i| < C_3 \log x (\log \log x + C_6) < 4.533 \times 10^{16}, \quad (32)$$

$$|e_j| < C_3 \log^2 x (\log \log x + C_6) / \log 2 < 4.761 \times 10^{19} \quad (33)$$

and similar upper bounds hold for $|h_k|$ and $|e_k|$, respectively. Combining these upper bounds with (27), we see that Λ has coefficients with absolute values at most 2.159×10^{36} . We use the LLL-reduction again with $C = 10^{111}$ and $\gamma = 2$, we obtain $|\Lambda| > 1.079 \times 10^{-75}$ and therefore $\log x < \log(7H/4) - \log |\Lambda| < 249.5$. This proves the lemma. \square

4 The final step

The final step is checking all possibilities of x . We note that from The Cunningham Project (see [15] or [5]), we know all prime factors of x 's below our upper bounds.

For $x = (a_i^{e_i} - 1)/(a_i - 1)$, we should check the residual orders of the other prime a_j modulo x . A summary is given in Tables 1-6, where Pn denotes a prime with n digits and (n) indicates that the residual order is a multiple of n . For example, putting $x = 2^{347} - 1 = pq$ with $p < q$, $o_q(3)$ is divisible by 6 since $q - 1$ is divisible by $2^3 \times 3^2$ and $3^{(q-1)/8}, 3^{(q-1)/3} \not\equiv 1 \pmod{q}$, although $3^{(q-1)/4} \equiv 1 \pmod{q}$, which yields that $(3^{e_2} - 1)/2 = p^{f_2} q^{g_2}$ with $g_2 > 0$ is impossible.

If $p = x = 2^{e_1} - 1$ is prime, then $e_1 \leq 511$ and therefore e_1 must belong to the set

$$\{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127\}.$$

Among them, there exists no e_1 such that the $o_x(3) = 1$, a prime or the square of a prime, as we can see from Table 1. Hence, by Lemma 2.3, we must have $(3^{e_2} - 1)/2 = q^{g_2}$. By Lemma 2.5, $(5^{e_3} - 1)/4$ must be divisible by $p = x$. Hence, by Lemma 2.3, $o_x(5) = 1$ or $o_x(5)$ must be a prime or a prime square and therefore, from Table 1, $e_1 = e_3 = 2$ or $e_1 = 5, e_3 = 3$. If $e_1 = e_3 = 2$, then $(5^{e_3} - 1)/4 = 6 = 2 \times 3$ and therefore $(3^{e_2} - 1)/2$ must be a power of 2, yielding that $e_2 = 2$. If $e_1 = 5$ and $e_3 = 3$, then $p = 31$ and $(3^{e_2} - 1)/2 = q^{g_2}$, yielding that $e_2 = 5$ or $(3^{e_2} - 1)/2 = q$.

If $x = 2^{e_1} - 1$ is not a prime power, then $e_1 \leq 359$ and therefore e_1 must belong to the set

$$\{4, 6, 9, 11, 23, 37, 41, 49, 59, 67, 83, 97, 101, 103, 109, 131, 137, 139, 149, \\ 167, 197, 199, 227, 241, 269, 271, 281, 293, 347\}.$$

Hence we can write $x = 2^{e_1} - 1 = pq$ for distinct primes $p < q$. By Lemma 2.5, $(3^{e_2} - 1)/2 = p^{g_2}$ and $(5^{e_3} - 1)/4 = p^{g_3}$ cannot simultaneously hold. In other words, at least one of these two integers must be divisible by q . But,

for no e_1 in the above set, $o_q(5)$ is 1, a prime or prime-square, as can be seen from Table 2. Hence $(3^{e_2} - 1)/2$ must be divisible by q . The only e_1 for which $o_q(3)$ is 1, a prime or prime-square is $e_1 = 4$. Then we must have $x = 2^4 - 1 = 3 \times 5$ and $(p, q) = (3, 5)$. But this implies that e_2 is divisible by 4 and $(3^{e_2} - 1)/2$ must be divisible by 2. Hence $(3^{e_2} - 1)/2$ cannot be of the form $p^{f_2}q^{g_2}$. Hence it cannot occur that $x = 2^{e_1} - 1$ is not a prime power.

If $x = (3^{e_2} - 1)/2 = p^{f_2}$ is prime or prime power, then

$$e_2 \in \{2, 3, 5, 7, 13, 71, 103\}.$$

For none of them, $o_p(2) = 1, 6$ or a prime power. Hence, as above cases, $(5^{e_3} - 1)/4$ must be divisible by p . Since $o_p(5)$ must be 1 or a prime power, we must have $e_2 \in \{2, 3, 5\}$. If $e_2 = 2$, then $p = 2$ and $e_3 = 2$, which yields that $q = 3$ and $e_1 = 2$. If $e_2 = 3$, then $p = 13$ and $e_3 = 4$, which is impossible since $(5^{e_3} - 1)/4 = 156 = 2^2 \times 3 \times 13$ has three distinct prime factors. If $e_2 = 5$, then $p = 11$ and $e_3 = 5$. Hence $(5^{e_3} - 1)/4 = 781 = 11 \times 71$. This implies that $2^{e_1} - 1 = 11^{f_1}71^{g_1}$, which is impossible since $2^{10} - 1 = 3 \times 31$ and $2^{35} - 1 = 31 \times 71 \times 127 \times 122921$.

If $x = (3^{e_2} - 1)/2$ is not a prime power, then

$$e_2 \in \{9, 11, 17, 19, 23, 37, 43, 59, 61, 223\}.$$

Hence we can write $x = (3^{e_1} - 1)/2 = pq$ for distinct primes $p < q$ with $p, q \neq 31$. However, $o_q(2)$ or $o_q(5)$ can never be 1, 6 or a prime power among the above e_2 's. Hence both $2^{e_1} - 1$ and $(5^{e_3} - 1)/4$ must be a power of p . By Lemma 2.5, we must have $p = 31$, which is impossible as mentioned above.

If $x = (5^{e_3} - 1)/4$ is a prime power, then

$$e_3 \in \{3, 7, 11, 13, 47, 127, 149, 181\}.$$

Among them, no e_3 gives a prime power (or one) residual order 3 (mod x) and only $e_3 = 3$ makes the residual order 2 (mod x) acceptable in view of Lemma 2.3. Hence $p = 31, e_3 = 3, e_1 = 5$ and $(3^{e_2} - 1)/2 = q^{f_2}$, which implies that $e_2 = 5$ or $(3^{e_2} - 1)/2 = q$.

If $x = (5^{e_3} - 1)/4$ is not a prime power, then

$$e_3 \in \{2, 5, 17, 23, 31, 41, 43, 59, 71\}.$$

Hence we can write $x = (5^{e_3} - 1)/4 = pq$ for distinct primes $p < q$. None of such $e_3 > 2$ gives an acceptable residual order 2 (mod q) or 3 (mod q) in view of Lemma 2.3. Hence we see that neither $2^{e_1} - 1$ nor $(3^{e_2} - 1)/2$ can be divisible by q and both must be a power of p , contrary to Lemma 2.5. Hence we must have $e_3 = 2, (p, q) = (2, 3)$. This yields that $e_1 = e_2 = 2$.

This completes the proof of Theorem 1.1.

Table 1: The residual orders of 3, 5 modulo p for $p = 2^{e_1} - 1$

e_1	$o_p(3)$	$o_p(5)$
2	N/A	2
3	6	6
5	30	3
7	126	42
13	910	1365
17	131070	65535
19	524286	74898
31	715827882	195225786
61	(10)	(15)
89	(6)	(84)
107	(6)	(6)
127	(6)	(6)

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Table 2: The residual orders of 3, 5 modulo p, q for $pq = 2^{e_1} - 1, p < q$

e_1	$2^{e_1} - 1 = pq$	$o_p(3)$	$o_q(3)$	$o_p(5)$	$o_q(5)$
9	7×73	6	12	6	72
11	23×89	11	88	22	44
23	47×178481	23	178480	46	44620
37	223×616318177	222	308159088	222	616318176
41	13367×164511353	6683	164511352	13366	164511352
49	$127 \times P13$	126	(8)	42	(8)
59	$179951 \times P13$	89975	(8)	89975	(8)
67	$193707721 \times P12$	96853860	(6)	8071155	(6)
83	$167 \times P23$	83	(10)	166	(166)
97	$11447 \times P26$	5723	(194)	11446	(194)
101	$P13 \times P14$	(303)	(303)	(303)	(303)
103	$2550183799 \times P22$	(166)	(206)	(249)	(309)
109	$745988807 \times P24$	(11663)	(118)	(214)	(118)
131	$263 \times P38$	131	(74)	262	(74)
137	$P20 \times P22$	(274)	(66290053)	(1202723)	(66290053)
139	$P13 \times P30$	(6)	(6)	(6)	(15)
149	$P20 \times P25$	(745)	(16)	(745)	(8)
167	$2349023 \times P44$	(26)	(22)	(26)	(22)
197	$7487 \times P56$	(3743)	(394)	(38)	(394)
199	$P12 \times P49$	(14)	(1393)	(8)	(1393)
227	$P18 \times P52$	(8)	(35)	(8)	(497)
241	$22000409 \times P66$	(8)	(5114261)	(482)	(5114261)
269	$13822297 \times P74$	(6)	(6)	(6)	(22)
271	$15242475217 \times P72$	(8)	(542)	(8)	(15)
281	$80929 \times P80$	(8)	(278)	(6)	(417)
293	$P26 \times P63$	(6)	(6)	(8)	(6)
347	$P23 \times P82$	(6)	(6)	(21)	(8)

Table 3: The residual orders of 2, 5 modulo p for $p^{f_2} = (3^{e_2} - 1)/2$

e_2	$o_p(2)$	$o_p(5)$
2	N/A	1
3	12	4
5	10	5
7	1092	364
13	398580	30660
71	(8)	(8)
103	(12)	(14)

Table 4: The residual orders of 2, 5 modulo p, q for $pq = (3^{e_2} - 1)/2, p < q$

e_2	$(3^{e_2} - 1)/2 = pq$	$o_p(2)$	$o_q(2)$	$o_p(5)$	$o_q(5)$
9	13×757	12	756	4	756
11	23×3851	11	3850	22	1925
17	1871×34511	935	595	935	3451
19	1597×363889	532	181944	532	22743
23	47×1001523179	23	(46)	46	(1073)
37	$13097927 \times P12$	(9731)	8594564351	(74)	(74)
43	$431 \times P18$	43	215	(22)	(22)
59	$14425532687 \times P18$	(3953)	(118)	(106)	(10679)
61	$603901 \times P24$	201300	(12)	150975	(145)
223	$P26 \times P81$	(446)	(12)	(6)	(446)

Table 5: The residual orders of 2, 3 modulo p for $p = (5^{e_3} - 1)/4$

e_3	$o_p(2)$	$o_p(3)$
3	5	30
7	6510	6510
11	1220703	369910
13	61035156	1211015
47	(94)	(6)
127	(18)	(18)
149	(10)	(6)
181	(12)	(15)

Table 6: The residual orders of 2, 3 modulo p, q for $pq = (5^{e_3} - 1)/4, p < q$

e_3	$(5^{e_3} - 1)/4 = pq$	$o_p(2)$	$o_q(2)$	$o_p(3)$	$o_q(3)$
2	2×3	N/A	2	1	N/A
5	11×71	10	35	5	35
17	409×466344409	204	3429003	204	116586102
23	$8971 \times P12$	8970	(8)	8970	2306995565
31	$1861 \times P18$	1860	(15)	310	(6)
41	$2238236249 \times P19$	279779531	(8)	(8)	(8)
43	$1644512641 \times P21$	(8)	(15)	(8)	(10)
59	$P17 \times P25$	(12)	(9)	(6)	(118)
71	$569 \times P47$	284	(142)	568	(452610863706241)

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